Patterns in Mathematics

Complete Solutions to all 'Figure It Out' Questions from the Chapter

1.1 What is Mathematics? — Figure It Out

1. Can you think of other examples where mathematics helps us in our everyday lives?

Yes — mathematics helps us in following a daily timetable, measuring ingredients while cooking, calculating money and discounts while shopping, keeping scores in games and sports, understanding rhythm and beats in music, reading traffic signal timings, and recognising patterns in nature such as petals of flowers, leaves, and honeycombs.

2. How has mathematics helped propel humanity forward? (Think of examples involving scientific experiments; running our economy and democracy; building bridges, houses or other complex structures; making TVs, mobile phones, computers, bicycles, trains, cars, planes, calendars, clocks, etc.)

Mathematics allows scientists to measure, record and analyse data accurately during experiments; it allows governments to run economies (budgets, taxes, trade) and democracies (counting votes, population data) fairly; it gives engineers the geometry and calculations needed to design safe bridges, houses and complex structures; and it provides the precise patterns and calculations used to manufacture technology such as TVs, mobile phones, computers, bicycles, trains, cars, planes, calendars and clocks.

1.2 Patterns in Numbers (Table 1) — Figure It Out

1. Can you recognise the pattern in each of the sequences in Table 1?

Yes. Each sequence in Table 1 grows by its own clear rule — see the rules listed in the next answer.

2. Rewrite each sequence of Table 1 along with the next three numbers, and write the rule for forming each sequence.

All 1's: 1,1,1,1,1,1,1,1,1,1,... — next three: 1, 1, 1. Rule: every term is 1.

Counting numbers: 1,2,3,4,5,6,7,8,9,10,... — next three: 8, 9, 10. Rule: add 1 to the previous term.

Odd numbers: 1,3,5,7,9,11,13,15,17,19,... — next three: 15, 17, 19. Rule: add 2 to the previous term, starting at 1.

Even numbers: 2,4,6,8,10,12,14,16,18,20,... — next three: 16, 18, 20. Rule: add 2 to the previous term, starting at 2.

Triangular numbers: 1,3,6,10,15,21,28,... — next three: 21, 28, 36. Rule: add the next counting number each time (+2, +3, +4, ...).

Squares: 1,4,9,16,25,36,49,64,81,100,... — next three: 64, 81, 100. Rule: nth term = $n \times n$.

Cubes: 1,8,27,64,125,216,... — next three: 343, 512, 729. Rule: nth term = $n \times n \times n$.

Virahānka numbers: 1,2,3,5,8,13,21,... — next three: 34, 55, 89. Rule: add the previous two terms.

Powers of 2: 1,2,4,8,16,32,64,... — next three: 128, 256, 512. Rule: multiply the previous term by 2.

Powers of 3: 1,3,9,27,81,243,729,... — next three: 2187, 6561, 19683. Rule: multiply the previous term by 3.

1.3 Visualising Number Sequences (Table 2) — Figure It Out

1. Copy the pictorial representations of Table 2 and draw the next picture for each sequence.

The next picture in each sequence would show: All 1's — 1 dot; Counting numbers — a staircase of 6 dots; Odd numbers — an L-shaped border of 11 dots; Even numbers — 12 dots; Squares — a 6×6 grid of 36 dots; Triangular numbers — a triangle with 21 dots; Cubes — a 4×4×4 solid cube of 64 unit cubes.

2. Why are 1, 3, 6, 10, 15, … called triangular numbers? Why are 1, 4, 9, 16, 25, … called square numbers? Why are 1, 8, 27, 64, 125, … called cubes?

They are called triangular numbers because that many dots can always be arranged in the shape of a triangle (rows of 1, 2, 3, 4, … dots). They are called square numbers because that many dots can be arranged in a perfect square grid (n rows × n columns). They are called cubes because that many unit cubes can be arranged into a perfect 3-dimensional cube (n × n × n).

3. You noticed that 36 is both a triangular number and a square number. Illustrate this with pictures.

36 is the 8th triangular number (1+2+3+…+8 = 36), so 36 dots can be arranged as a triangle with 8 rows. 36 is also 6^2, so the same 36 dots can be rearranged into a perfect 6×6 square grid. Drawing both the 8-row triangle and the 6×6 square using 36 dots each shows the same quantity taking two different shapes.

4. What would you call the sequence 1, 7, 19, 37, …? Draw these in your notebook. What is the next number in the sequence?

This is called the sequence of hexagonal numbers, because the dots can be arranged as a central dot surrounded by hexagonal rings. The next number after 37 is 61 (each hexagonal number follows the rule $1 + 3n(n-1)$; for $n = 5$, this gives $1 + 3 \times 5 \times 4 = 61$).

5. Can you think of pictorial ways to visualise the sequence of Powers of 2? Powers of 3?

Powers of 2 can be visualised as a row of dots that doubles at every stage (1, 2, 4, 8, 16, …), or as a branching tree where each branch splits into 2 new branches. Powers of 3 can be visualised similarly, where each branch splits into 3 new branches at every stage, or as a growing 3×3×3-style cube pattern.

1.4 Relations among Number Sequences — Try This / Figure It Out

1. Can you find a pictorial explanation for why adding counting numbers up and down (1, 1+2+1, 1+2+3+2+1, …) gives square numbers?

If we arrange dots so that the middle row (row n) is the longest and the rows above and below decrease symmetrically (1, 2, …, n, …, 2, 1 dots), the dots form a diamond/square shape tilted at 45° with n dots along each edge, and the total count is always n × n.

2. By imagining a large version of the picture, what is the value of 1 + 2 + 3 + … + 99 + 100 + 99 + … + 3 + 2 + 1?

This sum equals 100 × 100 = 10,000, since adding counting numbers up to n and back down always gives n^2.

3. Which sequence do you get when you add the All 1's sequence up? What sequence do you get when you add the All 1's sequence up and down?

Adding the All 1's sequence up (1, 1+1, 1+1+1, …) gives the Counting numbers (1, 2, 3, 4, …). Adding the All 1's sequence up and down gives the Odd numbers (1, 3, 5, 7, …).

4. Which sequence do you get when you start to add the counting numbers up? Give a pictorial explanation.

Adding the counting numbers up (1, 1+2, 1+2+3, 1+2+3+4, …) gives the Triangular numbers (1, 3, 6, 10, …). Pictorially, each new row of the staircase pattern adds one more dot than the row before it, building up a triangular arrangement.

5. What happens when you add up pairs of consecutive triangular numbers: 1+3, 3+6, 6+10, 10+15, …? Which sequence do you get? Explain with a picture.

You get the Square numbers: 4, 9, 16, 25, … This is because two consecutive triangular arrangements of dots (a triangle of side n and a triangle of side n+1) fit together perfectly to form a square grid of side (n+1) — this can be shown by rotating one triangle of dots and joining it to the other along their longest rows.

6. What happens when you start to add up powers of 2 starting with 1: 1, 1+2, 1+2+4, 1+2+4+8, …? Now add 1 to each of these numbers — what numbers do you get? Why does this happen?

The running sums are 1, 3, 7, 15, 31, … (always one less than the next power of 2). Adding 1 to each sum gives 2, 4, 8, 16, 32, … — the Powers of 2 sequence itself. This happens because $1+2+4+\ldots+2^{n-1} = 2^n - 1$ (a geometric series), so adding 1 always gives exactly 2^n, the next power of 2.

7. What happens when you multiply the triangular numbers by 6 and add 1? Which sequence do you get? Explain with a picture.

Multiplying a triangular number by 6 and adding 1 gives a hexagonal number: for example, 6×1+1=7, 6×3+1=19, 6×6+1=37, 6×10+1=61. This works because a hexagon can be built from 6 identical triangular wedges of dots arranged around a central dot, so 6 × (triangular number) + 1 (the centre) gives the total dots in the hexagonal pattern.

8. What happens when you start to add up hexagonal numbers: 1, 1+7, 1+7+19, 1+7+19+37, …? Which sequence do you get? Explain using a picture of a cube.

The running sums are 1, 8, 27, 64, … — the Cube numbers (1^3, 2^3, 3^3, 4^3, …)! Each hexagonal number can be thought of as one 2-D layer (cross-section) of unit cubes, and stacking these hexagonal layers one on top of another builds up a solid cube — which explains why the sum of hexagonal numbers gives cube numbers.

9. Find your own patterns or relations in and among the sequences in Table 1. Can you explain why they happen with a picture or otherwise?

Sample relation: the sum of the first n cube numbers equals the square of the nth triangular number, i.e., $1^3+2^3+3^3+\ldots+n^3 = [n(n+1)/2]^2$. For example, $1^3+2^3+3^3 = 1+8+27 = 36 = 6^2$, and 6 is indeed the 3rd triangular number. (Students may find and explain other valid relations from Table 1.)

1.5 Patterns in Shapes (Table 3) — Figure It Out

1. Can you recognise the pattern in each of the sequences in Table 3?

Yes — each shape sequence grows by a clear, describable rule, as explained in the next answer for every shape sequence.

2. Try to redraw each sequence in Table 3 and draw the next shape. Describe the rule for forming the shapes in each sequence.

Stacked Triangles: start with 1 small triangle; each next shape adds a new row of triangles below, growing into a bigger triangle.

Complete Graphs (K2, K3, K4, K5, K6): start with 2 points joined by 1 line; each next shape adds one more point, connected to every existing point by a new line.

Stacked Squares: start with 1 small square; each next shape adds another row and column of unit squares, forming a bigger square grid.

Koch Snowflake: start with an equilateral triangle (3 line segments); at every step, each straight segment is replaced by a 'speed bump' shape made of 4 smaller segments.

Regular Polygons: start with a triangle (3 sides); each next shape has one more equal side and angle — quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon, and so on.

1.6 Relation to Number Sequences — Figure It Out

1. Count the number of sides in each shape of the Regular Polygons sequence. Which number sequence do you get? What about the number of corners? Do you get the same sequence? Why?

The number of sides is 3, 4, 5, 6, 7, 8, 9, 10, … — the Counting numbers starting at 3. The number of corners follows the exact same sequence, because in any polygon the number of corners always equals the number of sides — each side connects two corners, and each corner joins two sides, so the two counts must always match.

2. Count the number of lines in each shape of the Complete Graphs sequence. Which number sequence do you get? Can you explain why?

K2, K3, K4, K5, K6 have 1, 3, 6, 10, 15 lines respectively — the Triangular number sequence. This happens because each time a new point is added, it must be joined by a new line to every point that already exists, so the total number of lines grows following the rule $n(n-1)/2$, which generates triangular numbers.

3. How many little squares are there in each shape of the Stacked Squares sequence? Which number sequence does this give? Can you explain why?

There are 1, 4, 9, 16, 25, … little squares — the Square number sequence, because the nth shape forms a perfect $n \times n$ grid of unit squares.

4. How many little triangles are there in each shape of the Stacked Triangles sequence? Which number sequence does this give? (Hint: how many triangles are there in each row?)

There are 1, 4, 9, 16, … little triangles — also the Square number sequence. Each row of the big triangle contains an odd number of small triangles pointing alternately up and down (1, 3, 5, 7, …), and since the sum of the first n odd numbers is always n^2, the total number of small triangles in the nth shape is n^2.

5. How many total line segments are there in each shape of the Koch Snowflake? What is the corresponding number sequence?

The number of line segments is 3, 12, 48, 192, … — that is, 3 times the Powers of 4 (3×4^0, 3×4^1, 3×4^2, …). This is because every one of the existing line segments is replaced by 4 new smaller segments at each step, so the total multiplies by 4 every time.